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An extended formulation approach to the edge-weighted maximal clique problem

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Abstract

We consider an extended formulation approach to the edge-weighted maximal clique problem. The problem is formulated by using additional variables for the set of nodes with the natural variables for the set of edges. We show that the proposed formulation is superior to the natural formulation both theoretically and practically. By using the projection technique, we can also derive new classes of facet-defining inequalities for the lower-dimensional polytope of the natural variables. Computational results are reported.

Keywords: Integer programming; Extended formulation; Cutting-plane/Facet-generation algorithm; Maximal Clique problem

1. Introduction

This paper considers the *weighted maximal b-clique problem* (MCPb), which can be stated as follows. Given a complete undirected graph $G=(V,E)$ with node weights $w_i \in \Re$, $i \in V$, edge weights $c_e \in \Re$, $e \in E$, and an integer b, (MCPb) finds a maximum weight clique with at most b nodes. The problem can be viewed as a generalization of the well-known maximum clique problem. By introducing variables x_i, for all $i \in V$ and y_e, for all $e \in E$, the problem can be formulated as the following 0–1 integer programming problem.

$$\max \sum_{i \in V} w_i x_i + \sum_{e \in E} c_e y_e$$

$$\text{s.t.} \sum_{i \in V} x_i \le b, \tag{1}$$

$$y_{ij} - x_i \le 0,\ y_{ij} - x_j \le 0, \quad \text{for all } (i,j) \in E \tag{2}$$

$$x_i + x_j - y_{ij} \le 1, \quad \text{for all } (i,j) \in E \tag{3}$$

$$x \in \{0,1\}^{|V|},\ y \in \{0,1\}^{|E|}.$$

Note that in the above formulation, the y variables can be treated as being continuous between 0 and 1.

In this paper, we are mainly interested in the special case of (MCPb) without node weights, that is, *the edge-weighted maximal clique problem.* There exist many applications of the problem, especially, in certain facility location problems, for example, see Späth, 1985, and Ravi et al., 1994. In this case, the problem can be formulated only by using the edge variables (natural formulation), see Dijkhuizen and Faigle, 1993. So the above formulation can be seen as an extended formulation of the problem. Though the node variables are not necessary in this case, the approach using the above formulation has many advantages over the natural formulation approach and this is the main theme of this paper.

Abbreviations: Edge-weighted maximal clique problem

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Dijkhuizen and Faigle, 1993, considered a strong cutting-plane approach to the edge-weighted maximal clique problem by using a natural formulation. However, they reported very poor performance of the proposed approach. In this paper, we present a new cutting-plane approach based on the extended formulation.

Suppose the constraint Eq. (1) is ignored, then the problem can be viewed as a linearized version of a boolean quadratic optimization problem, whose polyhedral structure has been studied by Padberg, 1989, and more generally, by Deza and Laurent, 1992a; Deza and Laurent, 1992b; De Simone, 1989, and Boros and Hammer, 1993. If the constraints Eq. (3) can be dropped (for example, when all edge weights are nonnegative), the problem reduces to a special case of the knapsack quadratic problem, which has been studied by Johnson et al., 1993. Hence we can use many known inequalities on those problems to devise a strong cutting-plane algorithm for (MCPb).

If we apply the *projection* to those inequalities onto the lower-dimensional space of the edge variables, we can obtain many interesting classes of facet-defining inequalities for the polytope associated with the natural formulation. Moreover, we show that the extended formulation with only one class of facet-defining inequalities gives a tighter LP relaxation than the natural formulation with several classes of facet-defining inequalities proposed by Dijkhuizen and Faigle, 1993.

The extended formulation approach was applied successfully to solve many hard integer programming problems, for example, see Eppen and Martin, 1987. In addition, the approach (with the projection) can be used to characterize the convex hull of certain combinatorial optimization problems, for example, see Balas and Pulleyblank, 1983.

Though we are mainly interested in (MCPb) without node weights in this paper, (MCPb), in itself, is a suitable model (or submodel) when we apply the column generation approach to certain complex clustering problems. In fact, the problem appears in a clustering problem arising in the broadband network design, see Park et al., 1994.

This paper is structured as follows. In Section 2, we review the previous study by Dijkhuizen and Faigle, 1993. In Section 3, we propose some classes of facet-defining inequalities of the polytope associated with the extended formulation. Comparisons of the two formulations using the projection technique are shown in Section 4. Some new facets of the polytope associated with the natural formulation are proposed in Section 5. In Section 6, we describe the cutting-plane algorithm for (MCPb). Computational results are given in Section 7. Finally, in Section 8, we give concluding remarks.

In the remainder of this section, we give some notations used in the paper. Let $p = |V|$ and $q = |E| = p(p-1)/2$. For any nonempty subset of nodes S, $\delta(S)$ denotes the set of edges with exactly one end in S. When S is a singleton, we write $\delta(i)$ instead of $\delta(\{i\})$. $E(S)$ is the set of edges with both ends in S. For two nonempty disjoint subsets S, T of V, $[S\colon T]$ is the set of edges with one end in S and the other end in T. Let ECP_b be the convex hull of the feasible solutions to the extended formulation. That is,

$$\mathrm{ECP}_b = \mathrm{conv}\{(x,y) \in \{0,1\}^{p+q} | (x,y) \text{ satisfies the first three equations}\}.$$

Also let CP_b be the convex hull of the feasible solutions to the natural formulation, that is, $\mathrm{CP}_b = \{\mathrm{conv}\{y \in \{0,1\}^q\} | y_e| = 1$ for all $e \in E(S) \wedge 0$ otherwise,
$\emptyset \neq S \subseteq V \wedge |S| \leq b\}$.

Then it is clear that CP_b can be obtained by projecting ECP_b onto the lower dimensional space of the edge variables.

2. Previous study on the polytope CP_b

In this section, we review the previous study on the edge-weighted maximal clique problem. Dijkhuizen and Faigle, 1993, studied the polyhedral structure of the polytope CP_b. In fact, they considered only the cliques whose number of nodes is greater than 1. But this doesn't alter any of the results in this paper and so, we simply ignore it. Their results are summarized in the following theorem.

Theorem 2.1. *The following inequalities are facet-defining for CP_b when $b \geq 4$.*

1. (Triangle Inequalities) *For any 3 distinct nodes i, j, k,*

$$y_{ik} + y_{jk} - y_{ij} \leq 1. \tag{4}$$

2. (Z-inequalities) *For any 4 distinct nodes i, j, l, m,*
$$y_{ij} + y_{jl} + y_{lm} - y_{il} - y_{jm} \leq 1. \tag{5}$$
3. (W-inequalities) *For any 5 distinct nodes* $i_1, \ldots, i_5$,
$$\sum_{e \in P} y_e - \sum_{e \in \bar{P}} y_e \leq 1, \tag{6}$$
where $P = \{(i_1,i_2),(i_3,i_4),(i_4,i_5)\}$ *and* $\bar{P} = \{(i_1,i_3),(i_2,i_4),(i_3,i_5)\}$.
4. (S-inequalities) *For any 6 distinct nodes* $i_1, \ldots, i_6$,
$$\sum_{e \in S} y_e = \sum_{e \in \bar{S}} y_e \leq 1, \tag{7}$$
where $S = \{(i_1,i_2),(i_2,i_3),(i_3,i_4),(i_4,i_5),(i_5,i_6)\}$ *and* $\bar{S} = \{(i_1,i)6\ ),(i_1,i_3),(i_2,i_4),(i_3,i_5),(i_4,i_6)\}$.
5. (Partition Inequalities) *For any nonempty subset S of V and a node* $t, \in V \setminus S$,
$$\sum_{i \in S} y_{it} - \sum_{e \in E(S)} y_e \leq 1. \tag{8}$$

The inequalities given in 1, 2, and 3 are special cases of the path inequalities, which are defined as follows (see Dijkhuizen and Faigle, 1993). Let $\{i_1, \ldots, i_k\}$ be the set of k distinct nodes and define

$$P = \{(i_1,i_2),(i_2,i_3),\ldots,(i_{k-1},i_k)\},$$
$$\bar{P} = (i_1,i_3),(i_2,i_4),\ldots,(i_{k-3},i_{k-1}),(i_{k-2},i_k).$$

Then the *path inequality* associated with $\{i_1, \ldots, i_k\}$ is defined as

$$\sum_{e \in P} y_e - \sum_{e \in \bar{P}} y_e \leq \frac{k+2}{4}.$$

They proposed a formulation of the problem as follows.

$$\max \sum_{e \in E} c_e y_e$$
$$\text{s.t.} \sum_{e \in \delta(v)} y_e \leq b - 1, \quad \text{for all } v \in V,$$

Eq. (4), Eq. (5),

$$y \in \{0,1\}^q.$$

Note that the above formulation requires q variables with $O(p^4)$ constraints, while the extended formulation shown in the previous section, needs $p + q$ variables with $O(p^2)$ constraints.

3. Facets of $\mathrm{ECP_b}$

In this section, we will introduce families of facet-defining inequalities for ECP_b. It can be easily shown that ECP_b is full-dimensional if and only if $b \geq 2$. In the following, we assume the condition holds. First, we will consider the trivial inequalities for ECP_b.

Proposition 3.1. *Let* $p \geq 3$.
1. *For all* $e \in E$, *the inequality* $y_e \geqslant 0$ *defines a facet of* ECP_b.
2. *For any two distinct nodes* $i,j \in V$, *the inequalities* $y_{ij} - x_i \leq 0$ and $x_i + x_j - y_{ij} \leq 1$ define facets of $\mathrm{ECP_b}$ if and only if $b \geq 3$.
3. *For any* $i \in V$, *neither the inequality* $x_i \leq 1$ *nor the inequality* $\sum_{e \in \delta(i)} y_e \leq b - 1$ *define a facet of* ECP_b.
4. *Neither the inequalities* $\sum_{i \in V} x_i \leq b$ *nor* $\sum_{e \in E} y_e \leq b(b-1)/2$ *define facets of* ECP_b.

Proof.
1. For sufficiency proof, see Padberg, 1989.
2. For sufficiency proof, see Padberg, 1989. Suppose $b = 2$. Then the following three inequalities are valid for ECP_b.
$$\sum_{e \in \delta(i)} y_e - x_i \leq 0, \tag{9}$$
$$x_i + x_j + x_k - y_{ij} - y_{ik} - y_{jk} \leq 1, \tag{10}$$
$$y_{ik} + y_{jk} - x_k \leq 0. \tag{11}$$
The inequality $y_{ij} - x_i \leq 0$ is dominated by Eq. (9). By adding the inequalities Eq. (10) and Eq. (11), we obtain the inequality $x_i + x_j - y_{ij} \leq 1$. Since any facet-defining inequality of the full-dimensional polyhedron is unique up to scalar multiplication, this shows the inequality is not facet-defining.
3. For the inequality $x_i \leq 1$, see Proposition 1 in Padberg, 1989. The inequality $\sum_{e \in \delta(i)} y_e \leq b - 1$ is dominated by the following valid inequality for ECP_b.
$$\sum_{e \in \delta(i)} y_e \leq (b-1) x_i.$$

4. Consider the following valid inequalities for ECP_b.

$$(b-1)\sum_{i\in V} x_i - \sum_{e\in E} y_e \le b(b-1)/2, \qquad (12)$$

$$\sum_{e\in\delta(i)} y_e - (b-1)x_i \le 0, \quad \text{for all } i \in V. \qquad (13)$$

By adding all the inequalities Eq. (12) and Eq. (13), we obtain the inequality $\sum_{e\in E} y_e \le b(b-1)/2$. Also by adding the inequality Eq. (12) multiplied by 2 and all of the inequalities in Eq. (13), we can obtain $\sum_{i\in V} x_i \le b$. □

As noted in the previous section, any valid inequalities of BQP (boolean quadric polytope, the polytope associated with the boolean quadratic optimization problem) is also valid for ECP_b. The following theorem introduces two classes of facet-defining inequalities of ECP_b which have been proved to be facet-defining for BQP by Padberg, 1989.

Theorem 3.2.

1. (Clique Inequality) *For given $S \subseteq V$ with $|S| \ge 3$ and an integer α, $1 \le \alpha \le |S|-2$, the clique inequality*

$$\alpha\sum_{i\in S} x_i - \sum_{e\in E(S)} y_e \le \alpha(\alpha+1)/2 \qquad (14)$$

defines a facet of ECP_b if and only if $S = V$ or $\alpha \le b-2$.

2. (Cut Inequality) *For any $S \subseteq V$ with $|S| \ge 1$ and $T \subseteq T/S$ with $|T| \ge 2$, the cut inequality*

$$\sum_{e\in[S:T]} y_e - \sum_{e\in E(S)} y_e - \sum_{e\in E(T)} y_e - \sum_{i\in S} x_i \le 0 \qquad (15)$$

defines a facet of ECP_b if and only if either $|S| = 1$ and $b \ge 3$ or $|S| \ge 2$ and $b \ge 4$.

Proof. For sufficiency, see Padberg, 1989. We will prove only the necessity part.

1. Suppose $S \ne V$ and $\alpha \ge b-1$. Then for any $t \in V \setminus S$, the following inequality is valid for ECP_b (see Theorem 3.3 2)).

$$\sum_{i\in S} y_{it} - \alpha x_t \le 0. \qquad (16)$$

Consider the following clique inequality on the set of nodes $S \cup \{t\}$.

$$\alpha \sum_{i\in S\cup\{t\}} x_i - \sum_{e\in E(S\cup\{t\})} y_e \le \alpha(\alpha+1)/2. \qquad (17)$$

Then by adding the inequalities Eq. (16) and Eq. (17), we obtain Eq. (14).

2. Suppose $|S| = 1$ and $b = 2$. The corresponding cut inequality Eq. (15) is dominated by the star inequality Eq. (13) with $i = s$, where $S = \{s\}$. Now suppose $|S| \ge 2$ and $b \le 3$. Then it can be easily shown the following inequality is valid and dominates Eq. (15).

$$\sum_{[e\in[S:T]} y_e - \sum_{e\in E(T)} y_e - \sum_{i\in S} x_i \le 0. \qquad \square$$

One may obtain more facet-defining inequalities of ECP_b by considering those of BQP proposed by Deza and Laurent, 1992a, Deza and Laurent, 1992b, and Boros and Hammer, 1993.

As discussed in the previous section, (MCPb) without the constraint Eq. (3) reduces to a special case of the knapsack quadratic problem (Johnson et al., 1993). A class of strong valid inequalities, called *the tree inequalities*, has been proposed for the problem by Johnson et al., 1993. They proved that the inequality defines a facet of the restricted polytope associated with the problem. Hence, it can be lifted to define a facet of the whole polytope by using the sequential lifting procedure (Nemhauser and Wolsey, 1988). But they also proved that the associated lifting problem is NP-hard. However, in the case of the current problem, we can show that the tree inequality, in itself, defines a facet of the whole polytope under a mild condition.

Theorem 3.3.

1. (Tree Inequality) *Let T be a tree of G with the set of nodes $V(T)$, $|V(T)| = b+1$, and the set of edges $F(T)$. Let $b \ge 3$. Then the inequality*

$$\sum_{e\in F(T)} y_e - \sum_{i\in V(T)} (d_i - 1)x_i \le 0 \qquad (18)$$

where d_i is the degree of the node i in T, defines a facet of ECP_b if and only if $b = p-1$ or T is *not a star.*

2. (Star Inequality) *For any node $r \in V$ the inequality,*

$$\sum_{e \in \delta(r)} y_e - (b-1)x_r \leq 0 \tag{19}$$

defines a facet of ECP_b if and only if $b \leq p$-1.

Sketch of the Proof

1. Suppose $b < p-1$ and T is a star with a center node r, then the corresponding tree inequality is dominated by the star inequality Eq. (19). To show sufficiency, suppose $b = p-1$ and T is a star, then the corresponding tree inequality reduces to the star inequality. Hence we assume $b \leq p-1$ and T is not a star. Let us define

$$P(T) = \text{conv}\{(x,y) \in ECP_b | x_i = 0, \\ \text{for all } i \in V \setminus V(T), \\ y_e = 0, \quad \text{for all } e \in E \setminus E(V(T))\}.$$

Note that $P(T)$ is obtained from ECP_b by fixing some variables to 0. Then we can show the tree inequality Eq. (18) defines a facet of $P(T)$ by the same method used in Johnson et al., 1993. So if we apply the sequential lifting procedure to the inequality, then we can obtain a facet-defining inequality for ECP_b. In the following, we will show that all of the lifting coefficients result in 0 if we apply the lifting procedure in a specified order. Let the lifting coefficient of the variable be x_i, be α_i, for $i \in V \setminus V(T)$ and that of y_e, be β_e, for $e \in E \setminus E(V(T))$.
When applying the lifting procedure to the inequality Eq. (18), first, we choose the variables x_i, for $i \in V \setminus V(T)$ and, then, y_e, for $e \in E(V \setminus V(T))$. Then it can be easily shown that $\alpha_i = 0$, for $i \in V \setminus V(T)$ and $\beta_e = 0$, for $e \in E(V \setminus V(T))$. Now the remaining variables are y_e, for $e \in [V(T){:}V \setminus V(T)]$. Apply the lifting procedure in the following order. Then by induction in step k, we can show all the lifting coefficients result in 0.
(Step 0) Set $T_1 = T$ and $k = 1$.
(Step k) If T_k is empty, stop. Otherwise, let L_k be the set of leaf nodes of T_k. Apply the lifting procedure to all of the variables y_e, for $e \in [L_k{:}V \setminus V(T)]$ (in an arbitrary order in this set). Set $T_{k+1} = T_k - L_k$ and $k = k+1$. Go to step k.

2. The proof is simple and so, it is omitted. □

If we use a lifting order different from the one used in the above proof, we can obtain a lifted tree inequality different from the original one, see Park et al., 1994. Johnson et al., 1993, generalized the tree inequality into the *forest inequality*. However, the inequality is of no interest in this case since the underlying graph is complete.

4. Comparisons of the two formulations

In this section, we will compare the two formulations of the edge-weighted maximal clique problem. Throughout this section, we use the projection of a polyhedron into a lower dimensional space. First, we will give the necessary notation and definitions.

Let us define

$$Q = \{(x,y) \in R^{p+q} | Ax + By \leq b, y \in \Psi\}.$$

Then the projection of the polyhedron Q into the y space, $\text{Pr}_y(Q)$, is defined as

$$\text{Pr}_y(Q) = \{y \in R^q | (x,y) \in Q, \quad \text{for some } x \in R^p\}.$$

Balas and Pulleyblank, 1983, proved the following result.

Theorem 4.1. *Let $W = \{w \in R_m | wA = 0, \quad w \geq 0\}$, where m is the number of rows of the matrix A. If $w^1, \ldots, w^s$ are the extreme rays of W, then*

$$\text{Pr}_y(Q) = \{y \in R^q | (w^k B) y \leq w^k b \\ \text{for all } k = 1, \ldots, s; y \in \Psi\}.$$

Let $Q_{m,n}$ be the set of feasible solutions to the following system of linear inequalities.

$$\sum_{t \in S} y_{it} - \sum_{e \in E(S)} y_e - x_i \leq 0, \quad \text{for all } S \subseteq V \setminus \{i\} \\ \text{with } 1 \leq |S| \leq m \text{ and for all } i \in V. \tag{20}$$

$$\sum_{i \in S} x_i - \sum_{e \in E(S)} y_e \leq 1, \quad \text{for all } S \subseteq V \\ \text{with } 1 \leq |S| \leq n. \tag{21}$$

$$y_e \geq 0, \quad \text{for all } e \in E. \tag{22}$$

Note that the inequalities Eq. (20) are special cases of the cut inequalities and Eq. (21) are those of the clique inequalities. Let us define $P_{m,n} = \Pr_y(Q_{m,n})$ and let

$$W_{m,n} = \left\{ \alpha \in \Re^r_+, \beta \in \Re^s_+ \middle| \sum_{S \subseteq V \setminus \{i\}, 1 \le |S| \le m} \alpha_{iS} - \sum_{i \in S \subseteq V, |S| \le n} \beta_s = 0, \text{for all } i \in V \right\},$$

where $r = p \sum_{k=1}^{m} {}_{(p-1)}C_k$, and $s = \sum_{k=1}^{n} {}_pC_k$.

In the following, we will characterize the set of extreme rays of $W_{m,n}$.

Proposition 4.2. *Let (α,β) be a ray of $W_{m,n}$. Then it is an extreme ray if and only if it is equivalent to the following form.*

$\beta_S = 1$ for some $S \subseteq V$, $1 \le |S| \le n$, and $\beta_T = 0$ otherwise

$\alpha_{iS_i} = 1$ for some $S_i \subseteq V \setminus \{i\}, 1 \le |S_i| \le m$, for all $i \in S$, and $\alpha_{iT} = 0$ otherwise.

Proof. The sufficiency of the condition can be proved easily. We prove only the necessity part. Let (α,β) be an extreme ray of $W_{m,n}$. Let us define $\Delta = \{T \subseteq V \mid \beta T > 0\}$ and $W = \cup\{T \subseteq V \mid T \in \Delta\}$. Then if $i \notin W$, $\alpha_{iS} = 0$ for all $S \subseteq V\setminus\{i\}, 1 \le |S| \le m$. Otherwise, if $i \in W$, there exists at least one S_i such that $S_i \subseteq V\setminus\{i\}, 1 \le |S_i| \le n$ and $\alpha_{iS_i} > 0$. Suppose more than one such subset exist. Let two distinct such subsets S_i and T_i be given. Let us define

$\alpha^1_{iS_i} = \alpha_{iS_i} + \alpha_{iT_i}$, $\alpha^1_{iT_i} = 0$, and $\alpha^1_{vT} = \alpha_{vT}$ otherwise,

$\beta^1_T = \beta_T$ for all $T \subseteq V$.

and

$\alpha^2_{iT_i} = \alpha_{iS_i} + \alpha_{iT_i}$, $\alpha^2_{iS_i} = 0$, and $\alpha^2_{vT} = \alpha_{vT}$ otherwise

$\beta^2_T = \beta_T$ for all $T \subseteq V$.

Then (α^1,β^1) and (α^2,β^2) are two rays of $W_{m,n}$ and (α,β) can be expressed as a positive combination of them, which contradicts the assumption that (α,β) is an extreme ray. Hence we can assume for each $i \in S$, there exists a unique subset S_i such that $S_i \subseteq V\setminus\{i\}$, $1 \le |S_i| \le n$ and $\alpha_{iS_i} > 0$.

Now suppose that $|\Delta| \ge 2$. Let us choose a subset $P \in \Delta$. Let us define

$\alpha^1_{iT} =$ β_P for all $i \in P$ and $T = S_i$; 0 otherwise

$\beta^1_T =$ β_P if $T = P$; 0 otherwise

and

$\alpha^2_{iT} =$ $\alpha_{iT} - \beta_P$ for all $i \in P$ and $T = S_i$; α_{iT} otherwise

$\beta^2_T =$ 0 if $T = P$; β_T otherwise.

Then (α^1,β^1) and (α^2,β^2) are two nonzero rays of $W_{m,n}$ and (α,β) an be expressed as a sum of them. Hence $|\Delta| = 1$ and this completes the proof. □

Therefore, using theorem 4.1, we obtain

Corollary 4.3.

$$P_{m,n} = \left\{ y \in \Re^q_+ \middle| \sum_{i \in S} \sum_{t \in S_i} y_{it} - \sum_{i \in S} \sum_{e \in E(S_i)} y_e - \sum_{e \in E(S)} y_e \le 1, \text{for all } S \subseteq V, 1 \le |S| \le n \text{ and } S_i \subseteq V\setminus\{i\}, 1 \le |S_i| \le m \text{ for all } i \in S. \right\}$$

Now we consider some interesting special cases of the above results. First, consider $Q_{1,2}$, which consists of the inequalities Eq. (2) and Eq. (3). Note that $Q_{1,2}$ with the cardinality constraint Eq. (1) gives an LP relaxation of the extended formulation.

Corollary 4.4. *$P_{1,2}$ is the set of feasible solutions to the following system of linear inequalities.*

$y_e \ge 0$, for all $e \in E$,

$$y_e \le 1, \quad \text{for all } e \in E, \tag{23}$$

$$y_{ik} + y_{jk} - y_{ij} \le 1, \quad \text{for all distinct three nodes } i,j,k \in V, \tag{24}$$

$$y_{ik} + y_{jl} - y_{ij} \le 1, \quad \text{for all distinct four nodes } i,j,k,l \in V. \tag{25}$$

The formulation $Q_{1,2}$ requires $p+q$ variables with $O(p^2)$ constraints and the formulation $P_{1,2}$ requires only q variables but $O(p^4)$ constraints.

Now we consider $Q_{2,2}$.

Corollary 4.5. *$P_{2,2}$ is the set of feasible solutions to the following system of linear inequalities.*

Eq. (22), Eq. (23), Eq. (24) and Eq. (25) and

$$y_{ik}+y_{jl}+y_{jm}-y_{ij}-y_{lm}\le 1,\quad \text{for all } i\ne j, l\ne m, j\ne l, j\ne m, i\ne k \tag{26}$$

$$y_{ik}+y_{il}+y_{jm}+y_{jn}-y_{ij}-y_{kl}-y_{mn}\le 1,\quad \text{for all } i\ne j, i\ne k, i\ne l, k\ne l, j\ne m, j\ne n, m\ne n. \tag{27}$$

Note that if $k=m$ in Eq. (26), the inequality corresponds to a Z-inequality. Also note that if $l=m$ in Eq. (27), the inequality is nothing but a W-inequality. The formulation $Q_{2,2}$ requires $p+q$ variables with $O(p^3)$ constraints and the formulation $P_{2,2}$ requires only q variables but $O(p^6)$ constraints. Hence if $Q_{2,2}$ with the cardinality constraint is used as an LP relaxation, the resulting upper bound will be tighter than that obtained using the natural formulation with all of the triangle, Z- and W-inequalities added.

The results in the above are sufficient to show the superiority of the extended formulation approach to the natural formulation approach. The extended formulation with a smaller number of constraints gives the same (or even tighter) linear programming relaxation than the natural formulation with a larger number of constraints.

5. New facets of CP_b

In this section, we propose new classes of facet-defining inequalities of the polytope CP_b, which are obtained or motivated by the projection of the polytope ECP_b (or BQP). First, we consider the inequality given in Corollary 4.3. When $m=p-1$ and $n=p$, the inequality can be rewritten as follows

$$\sum_{i\in S}\sum_{l\in S_i} y_{il}-\sum_{i\in S}\sum_{e\in E(S_i)} y_e-\sum_{e\in E(S)} y_e\le 1, \tag{28}$$

where $\emptyset\ne S\subseteq V$, $\emptyset\ne S_i\subseteq V\setminus\{i\}$ for all $i\in S$.

Let us define the inequality Eq. (28) *the sunflower inequality* if the following conditions hold.

1.

$$S_i\subseteq V\setminus S\quad \text{for all } i\in S$$

2.

$$\text{If } |S_i|=1\quad \text{for some } i\in S, \text{then } S_i\subseteq S_j \text{ for all } j\in S \tag{29}$$

3.

$$|S_i\cap S_j|\le 1\quad \text{for all distinct } i,j\in S.$$

We can show that all of the triangle, Z-, W- and partition inequalities are special cases of the *sunflower inequality*. The following theorem states that the inequality defines a facet of CP_b.

Theorem 5.1. *The sunflower inequality defines a facet of CP_b when $b\ge 4$.*

Proof. See Park et al., 1994. □

There exist other conditions under which the inequality Eq. (28) defines a facet of CP_b, see Park et al., 1994.

The inequality Eq. (28) does not always define a facet of CP_b. For example, for a nonempty subset of nodes $T\subseteq V\setminus S$, if $S_i=T$ for all $i\in S$, the corresponding inequality can be strengthened as follows.

$$\sum_{e\in[S:T]} y_e-\sum_{e\in E(S)} y_e-2\sum_{e\in E(T)} y_e\le 1 \tag{30}$$

Note that if $|S|\ge 3$, the inequality Eq. (30) dominates the inequality Eq. (28). Let us call the inequality Eq. (30) *2-partition inequality*. The following theorem states that the inequality is facet-defining for CP_b.

Theorem 5.2. *The 2-partition inequality Eq. (30) defines a facet of CP_b when $b\ge 4$.*

Proof. See Park et al., 1994. □

Finally, we mention that there may exist many other facets of CP_b that can be obtained by projecting down the inequalities of BQP. We think that this is particularly true if we consider the facets of BQP given in Deza and Laurent, 1992a; Deza and Laurent, 1992b) and others.

6. Cutting-plane algorithm

Now we will describe the cutting-plane algorithm for (MCPb) which uses the results in the previous sections. The algorithm is composed of two parts. In the first part, we solve the linear programming relaxation of the problem with the addition of the cutting-planes until we cannot find any more cutting-planes violated by the current solution. In the second part, if the final solution is not integral, we go into the branch-and-bound phase with the final formulation. Next, we will explain some details of the algorithm.

6.1. Initial formulation and cutting-planes

As an initial LP relaxation of (MCPb), we will use the following formulation.

$$(\mathrm{P1})\ \max \sum_{i\in V} w_i x_i + \sum_{e\in E} c_e y_e$$

$$\text{s.t. } y_{ij} - x_i \le 0, \quad y_{ij} - x_j \le 0, \quad \text{for all } (i,j) \in E$$

$$x_i + x_j - y_{ij} \le 1, \quad \text{for all } (i,j) \in E$$

$$\sum_{e\in\delta(i)} y_e - (b-1)x_i \le 0, \quad \text{for all } i \in V \tag{31}$$

$$\alpha \sum_{i\in V} x_i - \sum_{e\in E} y_e \le \alpha(\alpha+1)/2$$

$$\text{for all } \alpha = 1,\dots,p-2 \tag{32}$$

$$y_e \ge 0, \quad \text{for all } e \in E.$$

The inequalities Eq. (31) are the star inequalities and Eq. (32) are the clique inequalities defined on the set of nodes V. All of the inequalities used in (P1) are facet-defining for ECP_b under suitable conditions. Note that the above formulation gives a tighter LP relaxation than the formulation shown in Section 1. The number of constraints used in (P1) is $O(p^2)$.

To cut off the fractional solutions that may occur, we will use several classes of cutting-planes. First we define the clique inequality Eq. (14) with $|S| = 3$ and $\alpha = 1$ as the *triangle clique inequality*. Similarly the cut inequality Eq. (15) with $|S| = 1$ and $|T| = 2$ is called the *triangle cut inequality*. The following classes of inequalities are used as cutting-planes.

1. TS: = class of triangle clique inequalities,
2. TC: = class of triangle cut inequalities,
3. TR: = class of tree inequalities.

6.2. Separation algorithms

Now, we will describe the separation algorithms for the classes of cutting-planes given in the previous subsection. First, the separation problems for the inequalities in the classes TS and TC can be solved in $O(P^3)$ by simple comparisons. More specifically, for a given fractional solution (x^*, y^*) we generate MAXT number of most violated inequalities from each class, where MAXT is a prespecified number.

Now we consider the separation problem for the tree inequalities. First, suppose that a subset of nodes S with $|S| = b+1$ is given. Then we can show that the most violated tree inequality with $V(T) = S$ can be found by solving a minimum spanning tree problem on the complete graph on the set of nodes S.

Proposition 6.1. *For a given subset of nodes S with $|S| = b+1$, the most violated tree inequality with $V(T) = S$ can be found by solving a minimum spanning tree problem on the complete graph on the set of nodes S.*

Proof. Let $H = (S,F)$ be a complete graph on the set of nodes S. For each edge $(i,j) \in F$, let us define the edge weight $w_{ij} = x_i^* + x_j^* - y_{ij}$. Note that $w_e \ge 0$, for all $e \in F$. Let T be an arbitrary spanning tree in the graph H. Let us define $w(T) = \sum_{e\in E(T)} w_e$ and $a(T) = \sum_{e\in E(T)} y_e^* - \sum_{i\in S}(d_i - 1)x_i^*$, where d_i is the degree of the node i in the tree T, for all $i \in S$. Then it can be easily shown $-w(T) = a(T) - \sum_{i\in S} x_i^*$. Hence the most violated tree inequality, which corresponds to the spanning tree that maximizes $a(T)$, can be found by solving the minimum spanning tree problem on the graph H with edge weights w. □

Hence if $V(T)$ is fixed, we can easily find the most violated tree inequality. However, the general separation problem for the tree inequality seems to be very hard. We could not prove the NP-hardness for the separation problem. We can show that the problem can be formulated as a weighted k-cardinality tree problem (Fischetti et al., 1994) by using a similar transformation used in the above proposition. But the weighted k-cardinality clique problem is NP-hard (Fischetti et al., 1994).

We use a two phase method to find the violated tree inequality. The first phase is a greedy procedure to determine the set of nodes $V(T)$ with $|V(T)| = b + 1$ and the second is the minimum spanning tree problem as explained above.

6.3. Branch-and-bound phase

It can be easily shown that a given solution (x^*, y^*) is integral if and only if x^* is integral. Hence, before going to the branch-and-bound phase, only p node variables are set to be binary.

7. Computational results

To investigate the empirical performance of the extended formulation approach, we test the algorithm on instances generated randomly under the same setting used in Dijkhuizen and Faigle, 1993. We test the algorithm in two cases. In the first case, we generate the edge weights c_e in the range

$$0 \le c_e \le 1000.$$

In the second case, they are generated in the range

$$-500 \le c_e \le 500.$$

Dijkhuizen and Faigle, 1993 also tested their algorithm on the classical max-clique problem. However, we think that other approaches, for example, a cutting-plane algorithm for the node packing problem (Sigismondi, 1989), are more suitable in this case. So the case is not considered here.

For comparison with the Dijkhuizen and Faigle's approach, we also implemented their cutting-plane algorithm. In this case, we don't include the branch-and-bound phase since there are some difficulties in applying the off-the-shelf branch-and-bound procedure (the number of Z-inequalities is too large). For implementation details, see Dijkhuizen and Faigle, 1993.

For an LP solver and branch-and-bound routine, we use the CPLEX 3.0 callable mixed integer library. All tests are performed on a SUN/4 compatible workstation (25 MHz, 15.8 MIPs).

Table 1
Computational results for problems with nonnegative weights

p	b	# SOL	# CUT			# LP	LGAP (%)	IGAP (%)	TIME (sec)
			# TS	# TC	# TR				
10	3	10	54	98	17	17	2.3	0.0	5
	6	10	8	32	0	11	0.4	0.0	10
	9	10	0	0	0	10	0.0	0.0	8
15	4	10	286	620	60	41	5.5	0.04	8
	8	10	360	360	15	28	5.6	0.0	89
	12	10	0	0	0	10	0.0	0.06	4
20	5	9	652	2404	194	95	9.9	0.0	392
	10	10	1530	1530	80	61	13.7	0.0	448
	15	10	83	90	0	13	0.2	0.0	192
25	5	7	448	4238	432	105	8.5	0.3	969
	10	10	2952	3400	37	78	17.0	0.0	1471
	15	10	2025	2164	60	54	6.1	0.0	1296
	20	10	19	62	15	13	0.1	0.0	485
30	7	8	968	7912	552	172	11.8	0.1	7362
	14	8	581	937	125	230	21.0	0.3	9354
	21	10	2133	2903	252	66	2.3	0.0	2258
	28	10	0	0	0	10	0.0	0.0	252

7.1. Results with nonnegative weights

We generate 10 instances for each size (p,b) of the problem. Table 1 shows the computational results for the problems with nonnegative weights.

In the table, the headings # TS, # TC, and # TR represent the total numbers of the triangle clique inequalities, the triangle cut inequalities, and the tree inequalities generated over the 10 instances, respectively. The heading # SOL refers to the number of problems solved without invoking the branch-and-bound routine. The heading # LP refers to the total number of calls to LP solver. LGAP (IGAP, resp.) represents the (average) relative ratio between the objective value obtained by the first LP formulation (final LP formulations, resp.) and that of the integer optimum. Precisely, they are defined as follows.

$$\text{LGAP}(\%) = (\text{Initial LP} - \text{IP})/\text{IP} \times 100,$$

$$\text{IGAP}(\%) = (\text{Final Lp} - \text{IP})/\text{IP} \times 100.$$

The heading Time refers to the total CPU time needed to solve all 10 instances for each size.

Among 170 instances tested, only 10 instances need the branch-and-bound phase. However, even in this case, the number of branch-and-bound nodes generated does not exceed 4. When b is close to p, in most cases, we can obtain integer solutions only by solving the initial formulation (P1). Especially, when $p = 10$, $b = 9$, $p = 15$, $b = 12$, and $p = 30$, $b = 12$, integral solution is obtained by the initial formulation, for all of the instances tested. Hence the proposed initial formulation gives a very tight LP relaxation in this case.

The results show that (MCPb) becomes more difficult to solve when b is close to half of p. In this case, more cuts are generated and more time is needed to solve the problem. Also, in this case, LGAP is large.

7.2. Results with positive and negative weights

Table 2 shows the computational results for the problems with both positive and negative weights.

Similarly to the first case, only 10 instances need the branch-and-bound phase. Also in this case, at most 3 branch-and-bound nodes are generated. Note that LGAP is very significant compared to the first case. The initial formulation gives a very poor LP relaxation in this case, but the addition of cutting-planes results in integral solutions in most instances tested. In contrast to the first case, the problem with b very close to p is also difficult to solve in this case. This result seems to come from the fact that when the edge weights can have negative values, the

Table 2
Computational results for problems with positive and negative weights

p	b	# SOL	# CUT			# LP	LGAP (%)	IGAP (%)	TIME (sec)
			# TS	# TC	# TR				
10	3	10	20	80	4	15	2.8	0.0	5
	6	10	89	140	18	17	12.5	0.0	10
	9	10	122	260	10	22	24.7	0.0	12
15	4	10	249	400	40	30	48.5	0.0	36
	8	10	549	600	45	40	46.2	0.0	76
	12	10	518	790	39	50	48.9	0.0	60
20	5	10	586	1964	172	78	24.2	0.0	264
	10	10	1693	1920	50	74	67.4	0.0	506
	15	10	1531	2062	288	80	74.6	0.0	413
25	5	6	425	4618	431	108	22.6	0.2	918
	10	9	3342	4163	82	91	62.6	0.0	3169
	15	9	3196	4064	255	94	85.8	0.1	1932
	20	10	2935	3850	424	87	77.1	0.0	3747
30	7	6	1223	9025	634	203	37.5	0.8	4411
	14	10	5355	6000	93	130	97.4	0.0	4840
	21	10	5256	6800	321	146	99.9	0.0	4287
	28	10	4468	830	452	136	119.4	0.0	9248

cardinality constraint becomes very loose if b is sufficiently large compared to p.

7.3. Results on the Späth's data

Späth, 1985, considered the following facility location problem. Given a complete graph $G=(V,E)$ with nonnegative edge weights d_e, $e \in E$, find a clique on b nodes with the minimum edge weight. Let M be a strict upper bound on the edge weights d_e. Then by setting $w_e = M - d_e$ for all $e \in E$, the problem can be modeled as an edge-weighted maximal clique problem.

Späth, 1985, gives a set of data for $p = 25$ (and $M = 1000$). We tested our algorithm on the data and the results are shown in Table 3.

All of the instances are solved optimally approximately within 3 minutes. Among 23 instances, 4 instances ($b = 4,5,7,8$) need the branch-and-bound phase. Also in this case, at most 3 branch-and-bound nodes are generated. Dijkhuizen and Faigle, 1993, also tested their algorithm on the same data and they concluded the pure cutting-plane approach is not advisable. However, by using the extended formulation, we can solve the problems very efficiently.

Table 4
Computational results of D&F's algorithm ($p = 10$)

b	Nonnegative		Positive and nonnegative	
	# SOL	Time (sec)	# SOL	Time (sec)
3	7	373	4	380
6	8	318	7	228
9	8	95	8	317

7.4. Comparison with the Dijkhuizen and Faigle's algorithm

Table 4 shows the computational results obtained by applying the algorithm of Dijkhuizen and Faigle, 1993, on the same instances of size $p = 10$ which were used to test our algorithm. Those instances were solved by our algorithm within 2 seconds.

Among 60 instances tested, 18 instances are not solved by their algorithm. In addition, the CPU time is very significant since the separation routines need

Table 3
Computational results on the Späth's data ($p = 25$)

b	# CUT			# LP	LGAP (%)	IGAP (%)	Optimal value	Time (sec)
	#TS	#TC	#TR					
3	9	30	0	4	1.5	0.0	2853	23.1
4	10	50	0	6	2.6	0.6	5471	46.5
5	22	80	7	9	3.6	4.24	8478	89.5
6	74	230	13	24	8.5	0.0	12325	113.1
7	107	280	10	15	10.0	0.3	16559	145.2
8	40	60	5	4	12.5	0.7	21025	198.3
9	240	240	5	13	13.3	0.0	26235	160.0
10	260	260	4	14	15.2	0.0	31508	197.4
11	180	180	7	10	14.7	0.0	37834	137.2
12	220	220	5	12	14.8	0.0	44320	158.3
13	160	160	5	9	12.1	0.0	51089	127.6
14	140	140	4	8	8.6	0.0	57818	130.1
15	140	140	5	8	5.7	0.0	65110	110.2
16	100	100	3	6	3.2	0.0	73038	81.8
17	40	40	4	3	1.2	0.0	81660	49.7
18	20	20	0	2	0.1	0.0	90563	35.9
19	0	0	0	1	0.0	0.0	99767	24.4
20	0	0	0	1	0.0	0.0	109244	17.2
21	0	0	0	1	0.0	0.0	118167	18.1
22	0	0	0	1	0.0	0.0	127852	16.3
23	0	0	0	1	0.0	0.0	138097	11.4

much time. We have also tested their algorithm on the problem instances of $p = 15$. But the algorithm shows very poor performance, as observed by them (Dijkhuizen and Faigle, 1993). Moreover, the time to complete the addition of cutting-planes is very large, and in some cases, it even exceeds 1 hour.

8. Concluding remarks

In this paper, we proposed an extended formulation approach to the edge-weighted maximal clique problem. The proposed formulation has only $|V|$ additional variables with $|E|$ natural variables. As contrasted with the previous study by Dijkhuizen and Faigle, 1993, the approach with strong cutting-planes can solve instances of moderate size very efficiently.

By using the projection technique, we also compared two formulations and proposed new facet-defining inequalities for the polytope defined by natural variables. We show that the extended formulation with a few number of constraints gives a tighter LP relaxation than the natural formulation with a much larger number of constraints.

We think that the approach using an extended formulation combined with strong cutting-planes can be used as an efficient solution method for other combinatorial optimization problems, especially, when the pure cutting-plane approach gives a very poor performance.

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